

Math 223 - Multivariable Calculus
Practice Exam 1

Name: *solutions*

Please be sure to neatly **show and explain all of your work** and clearly label your answers. This is a closed-book, closed-notebook exam. Calculators are not allowed.

Please write and sign the Honor Pledge here when you are done:

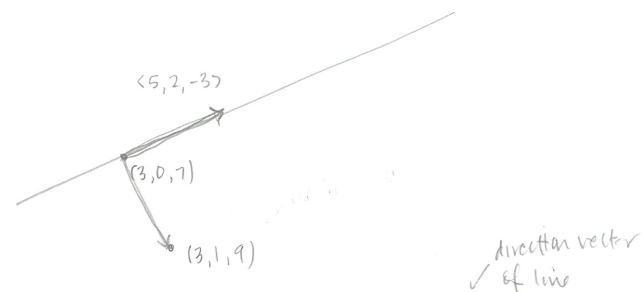
Signed:

Problem	Points
1	/12
2	/12
3	/12
4	/10
5	/12
6	/10
7	/12
Total	/80

1. Determine the equation of the plane containing the line

$$\vec{r}(t) = (3 + 5t, 2t, 7 - 3t)$$

and the point $(3, 1, 9)$. Show all work.



We have one vector in the plane: $\langle 5, 2, -3 \rangle = \vec{v}_1$

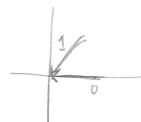
To get another consider the displacement vector
 $(3, 0, 7)$ to $(3, 1, 9)$: $\langle 0, 1, 2 \rangle = \vec{v}_2$

Then $\vec{v}_1 \times \vec{v}_2$ gives us a normal vector $\vec{n}$ to the plane:

using $(3, 1, 9)$
as a point in the plane

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 2 & -3 \\ 0 & 1 & 2 \end{vmatrix} = \langle 4 - (-3), 0 - 10, 5 - 0 \rangle = \langle 7, -10, 5 \rangle$$

The plane is given by $\langle 7, -10, 5 \rangle \cdot \langle x - 3, y - 1, z - 9 \rangle = 0$
 $\Rightarrow 7(x - 3) - 10(y - 1) + 5(z - 9) = 0 \Rightarrow \boxed{7x - 10y + 5z = 56}$



2. (a) Please compute the following limit or explain clearly why it does not exist.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{y}{x}$$

consider the line $y=0$: At every point on this line,
 $f(x,0) = \frac{0}{x} = 0$. So $\lim_{(x,0) \rightarrow (0,0)} \frac{0}{x} = \lim_{(x,0) \rightarrow (0,0)} 0 = 0$.

On the other hand, consider the line $y=x$. At every point on this line, $f(x,x) = \frac{x}{x} = 1$. So $\lim_{(x,x) \rightarrow (0,0)} \frac{x}{x} = \lim_{(x,x) \rightarrow (0,0)} 1 = 1$.

Since we get two different values as we approach $(0,0)$ along two different lines, the limit does not exist.

- (b) If $f(x,y,z) = e^{yz} \sin(xy)$, please compute f_{yz} .

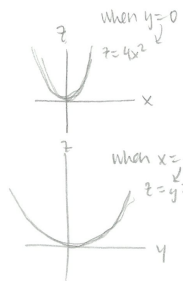
$$f(x,y,z) = e^{yz} \sin(xy)$$

$$f_y(x,y,z) = z e^{yz} \sin(xy) + e^{yz} \cos(xy) x \quad (\text{product rule})$$

$$= z e^{yz} \sin(xy) + x e^{yz} \cos(xy)$$

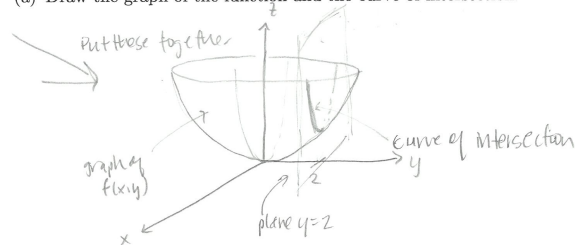
$$f_{yz}(x,y,z) = \left\{ \underbrace{e^{yz} \sin(xy)}_{\downarrow} + \underbrace{z y e^{yz} \sin(xy)}_{\downarrow} + \underbrace{x y e^{yz} \cos(xy)}_{\downarrow} \right\}$$

$$\text{Set } z = 4x^2 + y^2.$$



3. The graph of the function $f(x,y) = 4x^2 + y^2$ intersects the plane $y=2$ in a curve.

- (a) Draw the graph of the function and the curve of intersection.



- (b) Find the vector equation for the tangent line to the curve at the point $(1,2,8)$.

Note that along the curve of intersection, $y=2$. If we let $x=t$ we can parametrize by

$$\vec{r}(t) = \langle t, 2, 4t^2 + 4 \rangle$$

The point $(1,2,8)$ corresponds to $t=1$. This is a point on the tangent line to the curve. A direction vector for the line is given by $\vec{r}'(1)$.

$$\vec{r}'(t) = \langle 1, 0, 8t \rangle \quad \vec{r}'(1) = \langle 1, 0, 8 \rangle$$

Thus the line is given by $\underbrace{\langle 1, 2, 8 \rangle}_{\uparrow \text{pt}} + t \underbrace{\langle 1, 0, 8 \rangle}_{\uparrow \text{dir. vector}} = \underbrace{\langle 1+t, 2, 8+8t \rangle}_{\uparrow \text{telp.}}$

4. Please state whether each statement is true or false. If a statement is true, give an explanation why it is true. If it is false, give a counterexample or reason that shows that it is false.

(a) Suppose that $\vec{v}$, $\vec{x}$, and $\vec{w}$ are vectors in $\mathbb{R}^3$. If $\text{proj}_{\vec{w}} \vec{v} = \text{proj}_{\vec{w}} \vec{x}$, then $\vec{v} = \vec{x}$.

False: For example, if $\vec{w} = \langle 0, 1, 0 \rangle$, $\vec{v} = \langle 0, 1, 1 \rangle$, $\vec{x} = \langle 0, 1, 2 \rangle$, then

$\text{proj}_{\vec{w}} \vec{v} = \langle 0, 1, 0 \rangle$

$\left(\frac{\vec{w} \cdot \vec{v}}{\vec{w} \cdot \vec{w}} \right) \vec{w} = \frac{1}{1} \vec{w} = \vec{w}$

and $\text{proj}_{\vec{w}} \vec{x} = \langle 0, 1, 0 \rangle$

$\left(\frac{\vec{w} \cdot \vec{x}}{\vec{w} \cdot \vec{w}} \right) \vec{w} = \frac{1}{1} \vec{w} = \vec{w}$

equal even though $\vec{v} \neq \vec{x}$.

(b) Suppose that $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^3$. If $\vec{u} \times \vec{v} = \vec{0}$, then either $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$.

False: If $\vec{u} \times \vec{v} = \vec{0}$, this says that $\|\vec{u} \times \vec{v}\| = \|\vec{0}\| = 0$.

But $\|\vec{u} \times \vec{v}\|$ is the area of the parallelogram formed by $\vec{u}$ and $\vec{v}$. If $\vec{u}$ and $\vec{v}$ are parallel (scalar mults), but not necessarily $\vec{0}$, we would get $\vec{u} \times \vec{v} = \vec{0}$.

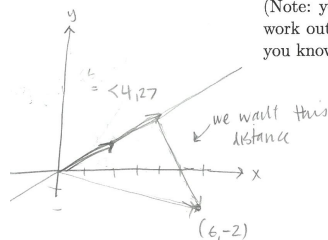


5. (a) Let $\vec{u} = \langle 6, -2 \rangle$ and $\vec{v} = \langle 2, 1 \rangle$. Compute $\text{proj}_{\vec{v}} \vec{u}$, the projection of $\vec{u}$ onto $\vec{v}$.

$$\begin{aligned} \text{proj}_{\vec{v}} \vec{u} &= \left(\frac{\vec{v} \cdot \vec{u}}{\vec{v} \cdot \vec{v}} \right) \vec{v} \\ &= \frac{(6)(2) + (-2)(1)}{(2)(2) + (1)(1)} \langle 2, 1 \rangle \\ &= \frac{10}{5} \langle 2, 1 \rangle \\ &= \langle 4, 2 \rangle \end{aligned}$$

- (b) Use your answer to Part (a) to find the distance from the point $(6, -2)$ to the line through the origin with direction vector $\langle 2, 1 \rangle$. Show all work and explain your reasoning.

(Note: you do not need to know a formula here. It is possible to work out this answer directly using the facts about vectors that you know.)



Let $\langle 6, -2 \rangle$ be the position vector of the point $(6, -2)$.

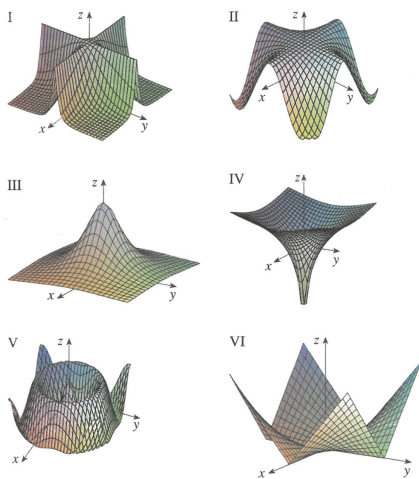
From above, the projection of $\langle 6, -2 \rangle$ onto the line through the origin with direction $\langle 2, 1 \rangle$ is given by $\langle 4, 2 \rangle$.

This means that $\langle 6, -2 \rangle - \langle 4, 2 \rangle = \langle 2, -4 \rangle$ is the vector through $(6, -2)$ and $\perp$ to the line. Then $\|\langle 2, -4 \rangle\|$ gives the distance btw the pt and the line: $\|\langle 2, -4 \rangle\| = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$

6. Please indicate which of the following represents a graph of

$$f(x, y) = \frac{1}{1+x^2y^2}.$$

Explain your reasoning.



If we consider the graph along the x-axis, i.e. $y=0$, we

get $f(x, 0) = \frac{1}{1+x^2(0)^2} = 1$ ← constant 1 on x-axis.

Similarly $f(0, y) = \frac{1}{1+(0)^2y^2} = 1$ ← constant 1 on y-axis.

This narrows the options to I or II.

Now, along $x=y$, we have $f(x, x) = \frac{1}{1+x^4}$. This decreases strictly as x increases. Thus, the graph must be option **I**.

Graph II does not have this behaviour.

7. At what point on the graph of

$$f(x, y) = x^2 + 3y^2$$

is the tangent plane parallel to the plane $4x - 6y + z = 7$?

Please show all work and explain your reasoning.

The normal vector to the tangent plane of the graph at (x, y) is given by $\langle f_x(x, y), f_y(x, y), -1 \rangle = \langle 2x, 6y, -1 \rangle$.

The normal vector to the plane $4x - 6y + z = 7$ is given by $\langle 4, -6, 1 \rangle$.

We are looking for the point where $\langle 2x, 6y, -1 \rangle \parallel \langle 4, -6, 1 \rangle$,

i.e. $\langle 2x, 6y, -1 \rangle = k \langle 4, -6, 1 \rangle$ for some k ;

$$\textcircled{1} \quad 2x = 4k$$

$$\textcircled{2} \quad 6y = -6k$$

$$\textcircled{3} \quad -1 = k \quad \leftarrow \text{from } \textcircled{3}, k = -1.$$

Then by $\textcircled{2}$, $6y = 6$ so $y = 1$

and by $\textcircled{1}$, $2x = -4$ so $x = -2$

Thus the point on the graph is $(-2, 1, (-2)^2 + 3(1)^2) = \boxed{(-2, 1, 7)}$